MATH 303 – Measure Theory
Mini-Project
Hausdorff Dimension

Instructions

- This document is divided into 4 sections. **Please submit one problem from each of the first 3 sections (3 total) for grading**. You are encouraged to work on additional exercises to more fully engage with the topic, but it is not required to write up any extra solutions, and there will not be any bonus points for submitting extra work.
- The problems are not all equally difficult; some are easier and others are more challenging. You are free to choose whichever problems you wish, but you are likely to learn more and have a more fulfilling experience if you attempt some of the harder exercises.
- Each problem will be graded out of 10 points and be counted with equal weight to a usual homework assignment.
- Please submit solutions as a single pdf with the exercises clearly numbered.
- Indicate the topic ("Hausdorff dimension") at the top of the first page.
- Please upload a pdf of your solutions by 23:59 on Monday, December 15.

The grade for this assignment will take into account both correctness and quality of presentation. More details on grading, as well as guidelines for mathematical writing, can be found on Moodle.

Self-similar geometric objects such as the Koch snowflake, Sierpiński carpet, and the middle-thirds Cantor set can be meaningfully assigned a notion of "dimension" that can take a non-integer value. How does one determine the dimension of a fractal object? There are several different approaches to dimension, but one of the most popular is the Hausdorff dimension, which relies on a family of measures that interpolate between the integer-dimensional Lebesgue measures. In this mini-project, you will develop the measure-theoretic tools to define Hausdorff dimension and compute dimensions of some famous examples.

Learning Objectives

After completing this mini-project, you will be able to:

- Reconstruct the proof of a long theorem consisting of many parts.
- Apply important theorems from the course (e.g., Riesz Representation Theorem and Fubini's Theorem) in a new context.
- Interpret dimension in terms of behavior under scaling.

1. Dimension as Scaling Exponent

When we say that a square is a two-dimensional shape, what do we mean? One reasonable interpretation is that a square is two-dimensional because when its side length is scaled by a factor r, the number of copies of the original square needed to cover the new square is r^2. That 2 in the exponent is telling us the dimension.

Example 1

Squares are two dimensional.

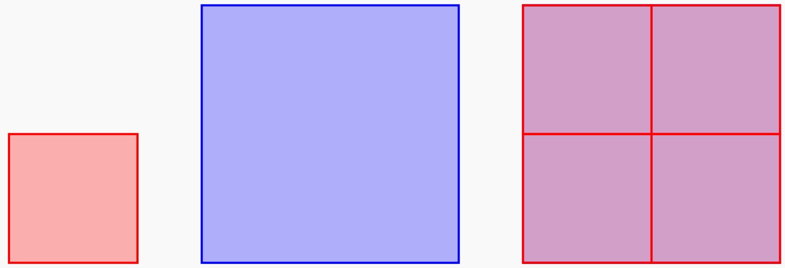

Figure 1. When the side length of a square is doubled, it covers a region equal to $4 = 2^{\mathbf{2}}$ of the original squares.

Triangles are also two dimensional.

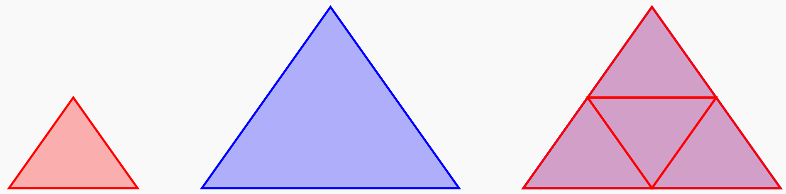

Figure 2. When the side length of a triangle is doubled, it covers a region equal to $4 = 2^{\mathbf{2}}$ of the original triangles.

Cubes are three dimensional.

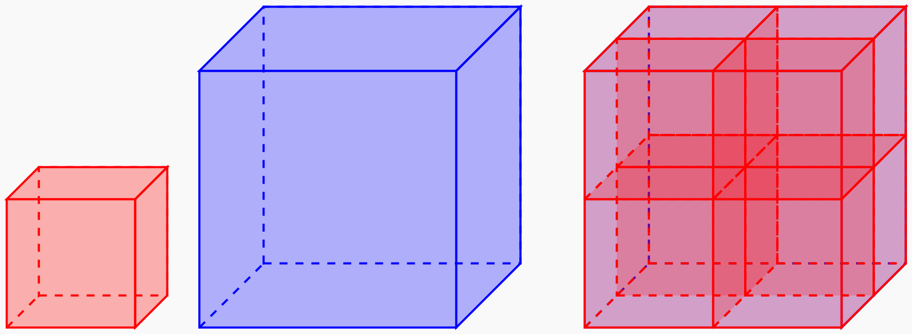

Figure 3. When the side length of a cube is doubled, it covers a region equal to $8 = 2^{\mathbf{3}}$ of the original cubes.

This intuitive conception of dimension already allows us to produce examples of self-similar fractal objects with non-integer dimension.

Example 2

The Cantor set has dimension $\frac{\log 2}{\log 3}$.

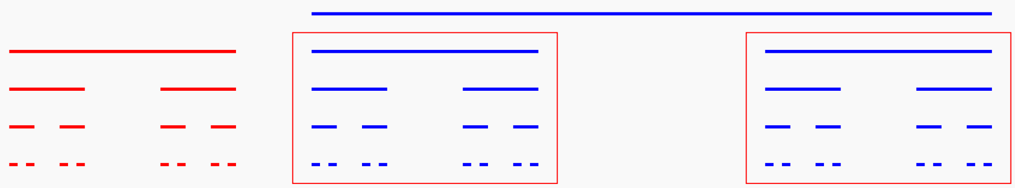

Figure 4. When the length of the middle-thirds Cantor set is scaled by 3, it covers $2 = 3^{\frac{\log 2}{\log 3}}$ copies of the original Cantor set.

Exercise 1

Determine the dimension of the Sierpiński carpet, pictured below. (Give a proof by picture similar to the previous examples.)

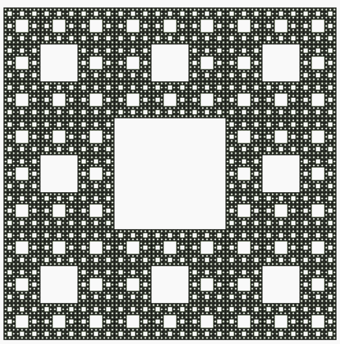

Figure 5. The Sierpiński carpet.

2. Hausdorff Measures

The interpretation of measure given so far is intuitively meaningful but lacks mathematical rigor and is ill-equipped for computing dimensions of irregular objects. We will now develop a measure-theoretic framework for treating dimension.

Motivated by the informal definition of dimension, we should expect that a d-dimensional object E has "size" proportional to $\operatorname{diam}(E)^d$ as the diameter varies. We thus adjust the definition of the Lebesgue measure to produce an outer measure with the appropriate scaling properties. Given $k \in \mathbb{N}$, $d \in [0, \infty)$, and $\delta > 0$, we define a function $\mathcal{H}_{k,\delta} : \mathscr{P}(\mathbb{R}^k) \to [0, \infty]$ by

$$\mathcal{H}^*_{d,\delta}(E) = \inf \left\{ \sum_{n=1}^{\infty} \operatorname{diam}(U_n)^d : E \subseteq \bigcup_{n=1}^{\infty} U_n, \operatorname{diam}(U_n) \leq \delta \right\}.$$

The stipulation that the sets U_n covering E have diameter $\operatorname{diam}(U_n) \leq \delta$ is required to accurately measure highly irregular objects.

Example 3

The curve $C_m = \{(x, \sin(mx)) : x \in [0,1]\}$ has length growing to ∞ as $m \to \infty$. However, if we cover C_m by the unit square $[0,1] \times [0,1]$, then we see that $\mathcal{H}_{1,\delta}(C_m) \le 1$ for $\delta \ge \sqrt{2}$. In order to account for each new oscillation in the curve as m grows to get a reasonable estimate of the length, we need to take δ on the order of $\frac{1}{m}$.

Exercise 2

Show that for every $E \subseteq \mathbb{R}^k$, the limit

$$\lim_{\delta \to 0^+} \mathcal{H}^*_{d,\delta}(E)$$

exists (as an extended real number).

Definition 4

Let $k \in \mathbb{N}$ and $d \in [0,\infty)$. The *d-dimensional Hausdorff outer measure* is the function $\mathcal{H}^*_d : \mathscr{P}(\mathbb{R}^k) \to [0,\infty]$ defined by

$$\mathcal{H}^*_d(E) = \lim_{\delta \to 0^+} \mathcal{H}_{d,\delta}(E).$$

Exercise 3

Prove that $\mathcal{H}^*_d$ is an outer measure.

Exercise 4

Show that Borel sets are $\mathcal{H}^*_d$-measurable. (Hint: Prove that closed sets are measurable.)

Definition 5

Let $k \in \mathbb{N}$ and $d \in [0,\infty)$. The *d-dimensional Hausdorff measure* is the Borel measure $\mathcal{H}_d = \mathcal{H}^*_d|_{\mathrm{Borel}(\mathbb{R}^k)}$.

Exercise 5

Show that $\mathcal{H}_d$ is isometry-invariant for every $d \ge 0$. Conclude that there is a constant $c_k \in (0,\infty)$ such that $\mathcal{H}_k = c_k \lambda_k$, where λ_k is the Lebesgue measure on $\mathbb{R}^k$.

Exercise 6

Show that $\mathcal{H}_0$ is equal to the counting measure.

Exercise 7

Let $E \subseteq \mathbb{R}^k$ be a Borel set.

(a) Suppose $\mathcal{H}_d(E) < \infty$. Prove that $\mathcal{H}_l(E) = 0$ for all $l > d$.
(b) Assume $\mathcal{H}_d(E) > 0$. Prove that $\mathcal{H}_l(E) = \infty$ for all $l < d$.

(c) Conclude that

$$\inf\{d \geq 0 : \mathcal{H}_d(E) = 0\} = \sup\{d \geq 0 : \mathcal{H}_d(E) = \infty\}.$$

The family of measures $(\mathcal{H}_d)_{d\geq 0}$ can be seen as interpolating between all of the integer-dimensional Lebesgue measures. We can use this family of Hausdorff measures to detect the scaling properties of a given set E.

Definition 6

The *Hausdorff dimension* of a Borel set $E \subseteq \mathbb{R}^k$ is

$$\dim_H(E) = \inf\{d \geq 0 : \mathcal{H}_d(E) = 0\} = \sup\{d \geq 0 : \mathcal{H}_d(E) = \infty\}.$$

3. Dimension of Self-Similar Sets

We will now see that the Hausdorff dimension appropriately captures the scaling properties of self-similar fractal objects such as the Cantor set and Sierpiński carpet.

Definition 7

Let $k \in \mathbb{N}$ and $r > 0$. A map $S : \mathbb{R}^k \to \mathbb{R}^k$ is a *similitude with scaling factor r* if $r^{-1}S$ is an isometry.

We can build self-similar fractals by a limiting process with similitudes. Suppose $\boldsymbol{S} = (S_1, \ldots, S_m)$ is a family of similitudes with scaling factor $r \in (0,1)$. For a set $E \subseteq \mathbb{R}^k$, define

$$\boldsymbol{S}(E) = \bigcup_{j=1}^{m} S_j(E).$$

We say that E is *$\boldsymbol{S}$-invariant* if $\boldsymbol{S}(E) = E$.

Example 8

The middle-thirds Cantor set is invariant for the pair of similitudes $\boldsymbol{S} = (S_1, S_2)$ with $S_1(x) = \frac{x}{3}$ and $S_2(x) = \frac{x}{3} + \frac{2}{3}$.

We call a nonempty bounded open set $U \subseteq \mathbb{R}^k$ a *separating set* for a family of similitudes $\boldsymbol{S} = (S_1, \ldots, S_m)$ if

$$\boldsymbol{S}(U) \subseteq U \qquad \text{and} \qquad S_i(U) \cap S_j(U) = \emptyset \quad (\forall i \neq j).$$

Example 9

For the similitudes in Example 8, the set $U = (0,1)$ is a separating set.

Proposition 10

Let $\boldsymbol{S} = (S_1, \ldots, S_m)$ be a family of similitudes with scaling factor $r < 1$. Suppose that $\boldsymbol{S}$ admits a separating set. Then there is a unique nonempty compact $\boldsymbol{S}$-invariant set.

Exercise 8

Prove Proposition 10 via the following steps.

(a) (Existence) Let U be a separating set. Show that $K = \bigcap_{n=0}^{\infty} \boldsymbol{S}^n(\overline{U})$ is nonempty, compact, and $\boldsymbol{S}$-invariant.

(b) (Uniqueness) Suppose X and Y are two nonempty compact $\boldsymbol{S}$-invariant sets. Let $d(X,Y) = \max_{x\in X}(\min_{y\in Y} d(x,y))$. Show that $d(X,Y) \leq r d(X,Y)$, and conclude that $X \subseteq Y$.

The next theorem gives precise mathematical meaning to the intuitive description of dimension from Section 1.

Theorem 11

Let $\boldsymbol{S} = (S_1, \ldots, S_m)$ be a family of similitudes with scaling factor $r < 1$. Suppose that $\boldsymbol{S}$ admits a separating set. Let $K \subseteq \mathbb{R}^k$ be the unique nonempty compact $\boldsymbol{S}$-invariant set. Then $\dim_H(K) = \frac{\log m}{\log(r^{-1})}$.

Exercise 9

(a) Prove that the middle thirds Cantor set has Hausdorff dimension $\frac{\log 2}{\log 3}$.

(b) Prove that the Sierpiński carpet has Hausdorff dimension equal to the number computed in Exercise 1.

The proof of Theorem 11 is rather lenghty. We break it into several parts, some of which are left as exercises.

Exercise 10

Let $\boldsymbol{S} = (S_1, \ldots, S_m)$ be a family of similitudes with scaling factor $r < 1$, and suppose that $\boldsymbol{S}$ admits a separating set. Let K be the unique nonempty compact $\boldsymbol{S}$-invariant set. Let $d = \frac{\log m}{\log(r^{-1})}$. Show that $\mathcal{H}_d(K) < \infty$.

Exercise 11

Let $\boldsymbol{S} = (S_1, \ldots, S_m)$ be a family of similitudes with scaling factor $r < 1$, and suppose $K \subseteq \mathbb{R}^k$ is a nonempty compact set that is invariant under $\boldsymbol{S}$. Pick a point $x \in K$, and consider the sequence of measures

$$\mu_n = \frac{1}{m^n} \sum_{i_1,\ldots,i_n=1}^{m} \delta_{S_{i_1}\circ\ldots\circ S_{i_n}(x)}.$$

(a) Show that $(\mu_n)_{n\in\mathbb{N}}$ is Cauchy in the vague topology so converges to a Borel probability measure μ.

(b) Show that μ has the following properties:

  (i) μ is $\boldsymbol{S}$-invariant in the sense that

$$\mu = \frac{1}{m} \sum_{i=1}^{m} (S_i)_* \mu.$$

  (ii) $\mu(K) = 1$.

(iii) If $U \subseteq \mathbb{R}^k$ is an open set and $U \cap K \neq \emptyset$, then $\mu(U) > 0$.

REMARK. The properties (ii) and (iii) together in part (b) of Exercise 11 say that the *support* of μ—the smallest closed set whose complement is μ-null—is equal to K.

PROOF OF THEOREM 11. Let $d = \frac{\log m}{\log(r^{-1})}$. We will show $0 < \mathcal{H}_d(K) < \infty$, from which it immediately follows that $\dim_H(K) = d$. The inequality $\mathcal{H}_d(K) < \infty$ is the content of Exercise 10, so we will work on showing that $\mathcal{H}_d(K) > 0$.

Choose $C > c > 0$ such that the separating set U is contained in a ball of radius C and contains a ball of radius cr^{-1}. Let $N = \left(\frac{1+2C}{c}\right)^k$. We will show $\mathcal{H}_d(K) \geq \frac{1}{N}$.

Let μ be the measure produced in Exercise 11.

CLAIM 1. If $B \subseteq \mathbb{R}^k$ is a ball of radius $\delta \in (0,1]$, then $\mu(B) \leq N\delta^d$.

Let $n \in \mathbb{N}$ such that $r^n < \delta \leq r^{n-1}$. By the $\boldsymbol{S}$-invariance of μ (property (i)), we have

$$\mu(B) = \frac{1}{m}\sum_{i=1}^{m} \mu(S_i^{-1}(B)) = \ldots = \frac{1}{m^n} \sum_{i_1,\ldots,i_n=1}^{m} \mu\left(\left(S_{i_1} \circ \ldots \circ S_{i_n}\right)^{-1}(B)\right).$$

The measure $\mu_{i_1,\ldots,i_n} = \left(S_{i_1} \circ \ldots \circ S_{i_n}\right)_* \mu$ has support $\left(S_{i_1} \circ \ldots \circ S_{i_n}\right)(K) \subseteq \left(S_{i_1} \circ \ldots \circ S_{i_n}\right)(\overline{U})$. Hence, $\mu\left(\left(S_{i_1} \circ \ldots \circ S_{i_n}\right)^{-1}(B)\right) = 0$ unless $B \cap \left(S_{i_1} \circ \ldots \circ S_{i_n}\right)(\overline{U}) \neq \emptyset$. The sets $U_{i_1,\ldots,i_n} = \left(S_{i_1} \circ \ldots \circ S_{i_n}\right)(U)$ are disjoint by the definition of a separating set, and each of them contains a ball of radius $r^n \cdot cr^{-1} = cr^{n-1} \geq c\delta$ by the choice of c and n. These two properties of the sets $U_{i_1,\ldots,i_n}$ restrict how many of the intersections $B \cap \overline{U}_{i_1,\ldots,i_n}$ can be nonempty. Indeed, suppose $B \cap \overline{U}_{i_1,\ldots,i_n} \neq \emptyset$ for some $i_1, \ldots i_n$. Then by the triangle inequality, the set $\overline{U}_{i_1,\ldots,i_n}$ is contained in the ball B' of radius $\delta + 2Cr^n < (1+2C)\delta$ with the same center as B. Thus,

$$\begin{aligned}\mu(B) = \frac{1}{m^n} \sum_{i_1,\ldots,i_n=1}^{m} \mu\left(\left(S_{i_1} \circ \ldots \circ S_{i_n}\right)^{-1}(B)\right) \\ \leq \frac{\#\left\{(i_1,\ldots,i_n) \in \{1,\ldots,m\}^n : B \cap \overline{U}_{i_1,\ldots,i_n} \neq \emptyset\right\}}{m^n} \\ \leq \frac{\#\left\{(i_1,\ldots,i_n) \in \{1,\ldots,m\}^n : U_{i_1,\ldots,i_n} \subseteq B'\right\}}{m^n}.\end{aligned}$$

On the other hand, since the sets $U_{i_1,\ldots,i_n}$ are disjoint and each contain a ball of radius $c\delta$, we have

$$((1+2C)\delta)^k \geq (c\delta)^k \cdot \#\left\{(i_1,\ldots,i_n) \in \{1,\ldots,m\}^n : U_{i_1,\ldots,i_n} \subseteq B'\right\}.$$

Therefore,

$$\mu(B) \leq \frac{1}{m^n}\left(\frac{1+2C}{c}\right)^k = Nr^{nd} < N\delta^d.$$

Suppose $(E_n)_{n\in\mathbb{N}}$ are bounded subsets of $\mathbb{R}^k$ such that $K \subseteq \bigcup_{n\in\mathbb{N}} E_n$. We want to show $\sum_{n=1}^{\infty} \operatorname{diam}(E_n)^d \geq \frac{1}{N}$. For each $n \in \mathbb{N}$, let B_n be a closed ball of radius $\delta_n = \operatorname{diam}(E_n)$ with

$E_n \subseteq B_n$. Then by the claim, $\mu(B_n) \leq N\delta_n^d$, so

$$\sum_{n=1}^{\infty} \delta_n^d \geq \frac{1}{N} \sum_{n=1}^{\infty} \mu(B_n) \geq \frac{1}{N}\mu(K) = \frac{1}{N}.$$

Thus, $\mathcal{H}_d(K) \geq \frac{1}{N}$. □

4. Some Applications of Hausdorff Dimension

Hausdorff dimension is a useful measure of "size" or "complexity" for sets of Lebesgue measure zero in Euclidean space. Null sets with positive Hausdorff dimension appear in many contexts in mathematics. In this section, we briefly described some results about dimension of sets arising from number theory and complex dynamics.

A classical area within number theory is the subject of Diophantine approximation, in which one analyzes approximations of real numbers by rational numbers. You proved two foundational results in Diophantine approximation in Homework 3, which are restated in the next two theorems.

Theorem 12: Dirichlet's Approximation Theorem

Given an irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$, there are infinitely many rational numbers $\frac{p}{q} \in \mathbb{Q}$ with $p \in \mathbb{Z}, q \in \mathbb{N}$ and $\gcd(p,q) = 1$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Notation. For $c > 0$, let $A_c \subseteq \mathbb{R}$ be the set of real numbers for which there are infinitely many rational approximations satisfying

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^c}.$$

Theorem 13: Khintchine's Approximation Theorem

If $c > 2$, then A_c is a set of Lebesgue measure zero.

The sets A_c have the nesting property $A_c \supseteq A_{c'}$ for $c' > c$. It also can be shown that for arbitrary c, the sets A_c, though of measure zero, are large in other senses: for example, they are all uncountable. The Hausdorff dimension quantifies the largeness of the sets A_c and reflects both the nesting property and their uncountability.

Theorem 14: Jarník–Besicovitch Theorem

For $c \geq 2$, $\dim_H(A_c) = \frac{2}{c}$.

Another set of interest in Diophantine approximation is the set of *badly approximable numbers*.

Definition 15

An irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$ is *badly approximable* if there is a constant $c > 0$ such that

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^2}$$

for every $\frac{p}{q} \in \mathbb{Q}$.

The set of badly approximable numbers is precisely the set of irrational numbers whose continued fraction expansion has bounded digits. An example of a badly approximable number is the golden ratio

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

The set of badly approximable numbers is small in terms of Lebesgue measure but large in the sense of Hausdorff dimension.

Theorem 16

The set of badly approximable numbers is a set of Lebesgue measure zero with Hausdorff dimension equal to 1.

The fact that badly approximable numbers form a set of measure zero was proved by Khintchine. The computation of the Hausdorff dimension was first done by Jarník; a later, highly influential, proof was given by Schmidt using what is now known as *Schmidt's game* to compute the Hausdorff dimension.

Another use of Hausdorff dimension is to quantify complexity in complex dynamics. Consider a polynomial map $f_c(z) = z^2 + c$ defined on the Riemann sphere[1] $\mathbb{C} \cup \{\infty\}$. Different points can have different behaviors under iteration of f_c. For example, if $c = -3$, then we have the following orbits under iteration:

$$0 \mapsto -3 \mapsto 6 \mapsto 33 \mapsto \dots$$

$$1 \mapsto -2 \mapsto 1 \mapsto -2 \mapsto \dots$$

Here, the first orbit diverges to infinity, while the second is periodic. The orbit beginning at $z = 2$,

$$2 \mapsto 1 \mapsto -2 \mapsto 1 \mapsto -2 \mapsto \dots$$

is pre-periodic. The map f_c also has two fixed points, the roots of the quadratic polynomial $z^2 - z - 3$, which are $\frac{1}{2} \pm \frac{\sqrt{13}}{2}$.

Definition 17

The *filled Julia set* K_c of the quadratic map f_c is the set of points with bounded orbit under f_c, i.e.

$$K_c = \left\{ z \in \mathbb{C} : \sup_{n \in \mathbb{N}} |f_c^{\circ n}(z)| < \infty \right\},$$

where $f_c^{\circ n} = \underbrace{f_c \circ \dots \circ f_c}_{n \text{ times}}$. The *Julia set* J_c is the boundary of K_c.

Points outside of the filled Julia set have well-understood behavior: they all escape to infinity under iteration. Points in the interior of the filled Julia set also have predictable behavior and stay confined to the interior. The Julia set is a compact set that is invariant under iterations and captures the interesting, chaotic dynamical behavior of the map f_c.

[1]The Riemann sphere is the one-point compactification of the complex plane constructed by "adding a point at infinity." In most of this discussion, you can think about f_c as a map just of the complex plane, but for more general maps, it is useful for the dynamics to be considered on a compact space for various technical reasons.

Julia sets are a source of beautiful fractal images. If you have not seen them before, there is a nice collection of Julia sets for different values of c on the wikipedia page: https://en.wikipedia.org/wiki/Julia_set#Quadratic_polynomials.

In the context of complex dynamics, the Hausdorff dimension of the Julia set therefore serves as a meaningful measurement of the complexity of the dynamics of iterations of f_c. There is no general formula for $\dim_H(J_c)$ in terms of c, and it can be extremely difficult to compute the dimension outside of a limited number of examples. However, there are several interesting results related to dimensions of Julia sets.

Definition 18

The *Mandelbrot set* is the set of $c \in \mathbb{C}$ such that the corresponding Julia set J_c is connected.

Two of the most significant results about Hausdorff dimension related to Julia sets are due to Shishikura in the 1990s.

Theorem 19

The boundary of the Mandelbrot set has Hausdorff dimension 2. Moreover, for a generic[a] point c in the boundary of the Mandelbrot set, $\dim_H(J_c) = 2$.

[a]The meaning of the word "generic" here is in the topological sense of Baire category. If you are not familiar with this notion, it is useful to know that one of the implications of being generic is that the set of points c with this property is dense in the boundary.